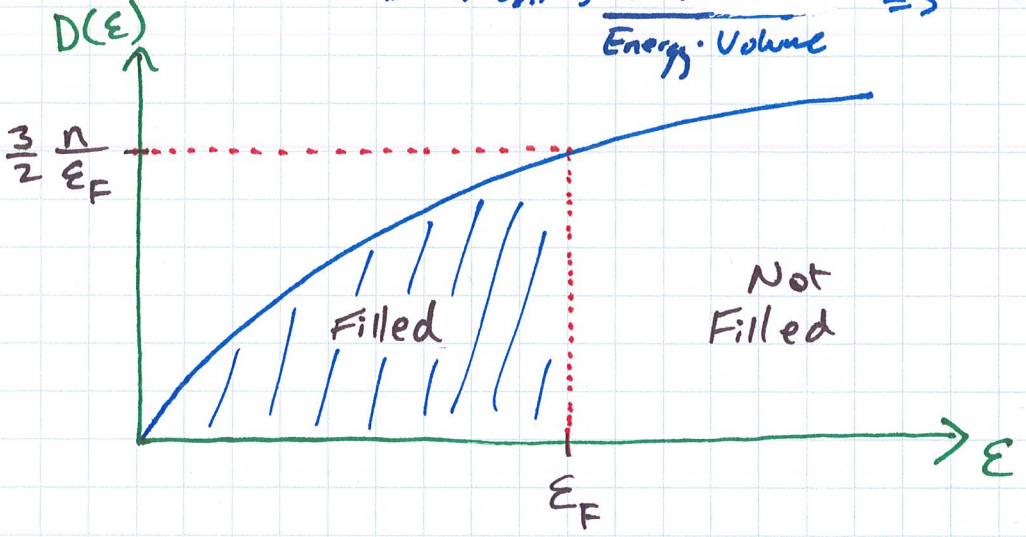
Note: $D(\varepsilon_F) = \frac{3}{2}\frac{n}{\varepsilon_F} = \frac{m k_F}{\hbar^2 \pi^2}$

$D(\varepsilon) = D(\varepsilon_F)\sqrt{\frac{\varepsilon}{\varepsilon_F}}$

$D(\varepsilon)$ units : $\frac{1}{\text{Energy} \cdot \text{Volume}}$ $\Rightarrow$ $J^{-1} \cdot m^{-3}$




## Quantum Free $\bar{e}$ (Sommerfeld) @ finite temperatures

→ moving to $T > 0$ means $\bar{e}$ will fill our single $\bar{e}$ allowed states in a slightly different manner

Grand canonical ensemble:

→ expected single-state occupancy $\langle N_i \rangle \Rightarrow \left( f(\varepsilon_k) \right)$

$$\langle N_i \rangle = k_B T \frac{1}{Z} \left( \frac{\partial Z}{\partial \mu} \right)_{V,T}$$

$\mu \rightarrow$ chemical potential

$Z \rightarrow$ partition function

For any state (allowed $k$) the partition function is given by a sum over all allowed occupancies ($N_i$): i.e. microstates.

$$\mathcal{Z} = \sum_i e^{(N_i\mu - E_i)/k_BT}$$

For $\bar{e}$'s (Fermions), there are only two microstates:

$$N_i = 0 \quad , \therefore E_i = 0$$

$$N_i = 1 \quad , \therefore E_i = \varepsilon_i$$

$$\therefore \quad \mathcal{Z} = e^0 + e^{(\mu-\varepsilon_i)/k_BT}$$

$$\mathcal{Z} = 1 + \exp\left[(\mu-\varepsilon_i)/k_BT\right]$$

$\therefore$ For state $i$:

$$\langle N_i \rangle = k_BT \left( \frac{1}{1+\exp\left[(\mu-\varepsilon_i)/k_BT\right]} \right)$$

$$\times \frac{\partial}{\partial \mu} \left( 1 + \exp\left[(\mu-\varepsilon_i)/k_BT\right] \right)_{V,T}$$

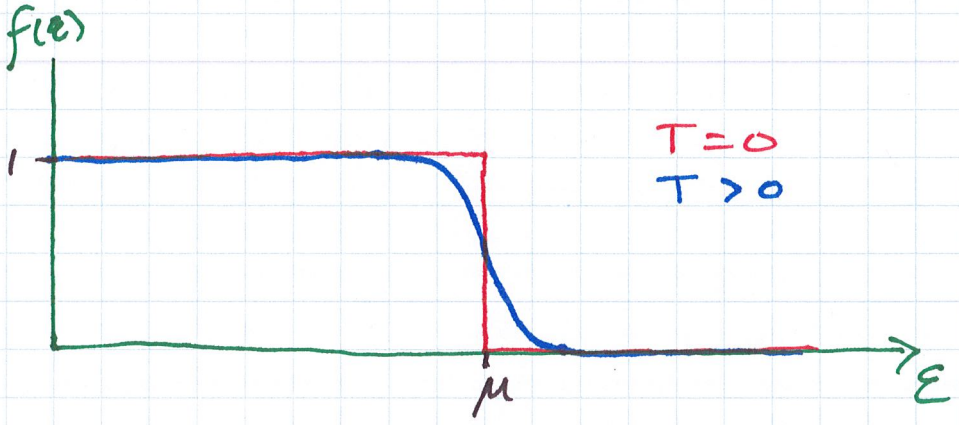
$$\langle N_i \rangle = \frac{1}{\exp\left[(\varepsilon_i-\mu)/k_BT\right] + 1}$$

We can drop the index & write as a continuous function.

$$\langle N \rangle = f(\epsilon) = \frac{1}{\exp[(\epsilon-\mu)/k_B T] + 1}$$

↓

Fermi-Dirac distribution




Previously:

$$@\ T=0 \qquad f(\epsilon) = \begin{cases} 1, & \epsilon < \epsilon_F \\ 0, & \epsilon > \epsilon_F \end{cases}$$

$$\therefore \quad \lim_{T\to 0} \mu = \epsilon_F \qquad \text{in general} \Rightarrow \mu = \mu(T)$$

$\epsilon_F = \mu$ is not strictly true @ $T > 0$, but it is a good approximation even up to room temperature

Recall: chemical potential $\mu_i = \left.\frac{\partial U}{\partial N_i}\right|_{S,V,N_{j\neq i}}$

is the energy cost to add/remove a particle to/from the system/ensemble.

What is the energy of this state @ finite T?

Same approach as T=0:

$$u = \frac{U}{V} = \frac{2}{V}\sum_{\vec{k}} \varepsilon_{\vec{k}} f(\varepsilon_{\vec{k}})$$

(2 → 2 $e^-$'s per $\vec{k}$ ($\uparrow\downarrow$))

- $u$: energy density
- $\varepsilon_{\vec{k}}$: energy of state $\vec{k}$
- $f(\varepsilon_{\vec{k}})$: expected occupancy of state $\vec{k}$ (FD-dist.)

using same trick: $\frac{1}{V}\sum_{\vec{k}} F(\vec{k}) = \int \frac{d^3\vec{k}}{8\pi^3} F(\vec{k})$

$$u = \int \frac{d^3\vec{k}}{4\pi^3} \varepsilon_{\vec{k}} f(\varepsilon_{\vec{k}})$$

→ both $\varepsilon_{\vec{k}}$ & $f(\varepsilon_{\vec{k}})$ are functions of $\vec{k}$.

Note from the dfn of DOS => $D(\varepsilon)$, $u$ can be expressed as:

$$\boxed{u = \int D(\varepsilon)\,\varepsilon\, f(\varepsilon)\, d\varepsilon}$$

→ drop $\vec{k}$ label since $\int d\varepsilon$ now.

likewise: $n = \frac{N}{V} = \frac{2}{V}\sum_{\vec{k}} f(\varepsilon_{\vec{k}})$

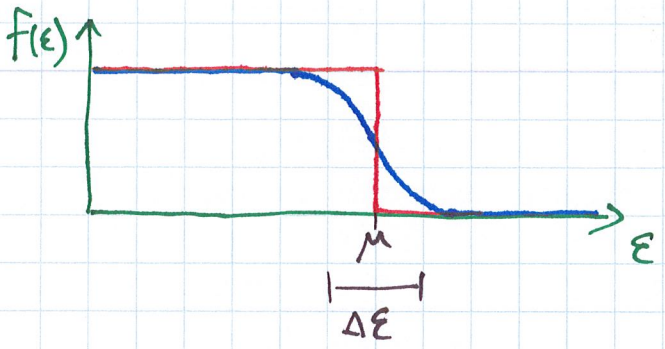
$$\boxed{n = \int D(\varepsilon) f(\varepsilon)\, d\varepsilon}$$

Note: $D(\varepsilon) \propto \sqrt{\varepsilon}$

These integrals have no closed form (try in your fav. solver).

and, in general, $D(\varepsilon)$ may be much more complicated than the $\sqrt{\varepsilon}$ form....

BUT




$f(T=0)$ & $f(T>0)$ only differ by the small region around $\varepsilon = \mu$.

∴ if $T \ll T_F$ => check this condition for yourself.

We can approximate the integrals by the first couple terms of an expansion about $\varepsilon = \mu$:

Sommerfeld Expansion:

for a function of $\varepsilon$, $H(\varepsilon)$:

$$\int_0^\infty H(\varepsilon) f(\varepsilon)\, d\varepsilon = \int_0^\mu H(\varepsilon)\, d\varepsilon + H'(\mu)\frac{\pi^2}{6}(k_B T)^2 + O(k_B T)^4 + \ldots$$

Back to $u$ & $n$: $D(\varepsilon) = D(\varepsilon_F)\sqrt{\frac{\varepsilon}{\varepsilon_F}}$

$$u = \frac{D(\varepsilon_F)}{\sqrt{\varepsilon_F}} \int \varepsilon^{3/2} f(\varepsilon)\, d\varepsilon \quad \Rightarrow \quad H(\varepsilon) = \varepsilon^{3/2}$$

$$n = \frac{D(\varepsilon_F)}{\sqrt{\varepsilon_F}} \int \varepsilon^{1/2} f(\varepsilon)\, d\varepsilon \quad \Rightarrow \quad H(\varepsilon) = \varepsilon^{1/2}$$

Apply S.E.:

$$u \approx \frac{D(\varepsilon_F)}{\sqrt{\varepsilon_F}} \left[ \int_0^{\mu} \varepsilon^{3/2}\, d\varepsilon + \frac{3}{2} \varepsilon^{1/2} \Big|_{\mu} \frac{\pi^2}{6} (k_B T)^2 \right]$$

$$n \approx \frac{D(\varepsilon_F)}{\sqrt{\varepsilon_F}} \int_0^{\mu} \varepsilon^{1/2}\, d\varepsilon + \frac{1}{2} \varepsilon^{-1/2} \Big|_{\mu} \frac{\pi^2}{6} (k_B T)^2 \Big]$$

$$\therefore \; u \approx \frac{D(\varepsilon_F)}{\sqrt{\varepsilon_F}} \left[ \frac{2}{5} \mu^{5/2} + \frac{3}{2} \mu^{1/2} \frac{\pi^2}{6} (k_B T)^2 \right]$$

$$n \approx \frac{D(\varepsilon_F)}{\sqrt{\varepsilon_F}} \left[ \frac{2}{3} \mu^{3/2} + \frac{1}{2} \mu^{-1/2} \frac{\pi^2}{6} (k_B T)^2 \right]$$

Recall: $D(\varepsilon_F) = \frac{3}{2} \frac{n}{\varepsilon_F}$ and $n \neq n(T)$ (constant volume).

$$\therefore \quad n = \frac{3}{2}\frac{n}{\varepsilon_F^{3/2}}\left[\frac{2}{3}\mu^{3/2} + \frac{1}{2}\mu^{-1/2}\frac{\pi^2}{6}(k_B T)^2\right]$$

$$\therefore \quad \frac{2}{3}\varepsilon_F^{3/2} = \frac{2}{3}\mu^{3/2}\left[1 + \frac{3}{4}\frac{\pi^2}{6}\left(\frac{k_B T}{\mu}\right)^2\right]$$

$$\text{or} \quad \mu = \varepsilon_F\left[\frac{1}{1 + \frac{3}{4}\frac{\pi^2}{6}\left(\frac{k_B T}{\mu}\right)^2}\right]^{2/3}$$

assume $\left(\frac{k_B T}{\mu}\right)^2 \ll 1$

and $\left(\frac{k_B T}{\mu}\right)^2 \approx \left(\frac{k_B T}{\varepsilon_F}\right)^2$

$$\mu = \varepsilon_F\left(1 - \frac{2}{3}\frac{3}{4}\frac{\pi^2}{6}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right)$$

$$\therefore \quad \mu = \varepsilon_F + \Delta\mu$$

where $\frac{\Delta\mu}{\varepsilon_F} \approx -\frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2$

For $T > 0$ but $T \ll T_F$, $\mu$ differs by $\varepsilon_F$ on the order of $\left(\frac{k_B T}{\varepsilon_F}\right)^2$ per (fractional)

$\rightarrow = \left(\frac{T}{T_F}\right)^2$

Recall: $T_F \approx 10^5$ K

room temp: $T \approx 3 \times 10^2$ K

$\Rightarrow$ large range of validity.

Likewise, we find:

$$u = u_0 + D(\varepsilon_F)\frac{\pi^2}{6}(k_B T)^2 = u_0 - 3\Delta\mu n$$

where $u_0$ is the g.s. ($T=0$) energy: $u_0 = \frac{3}{5}\varepsilon_F n$

## Heat capacity

→ Now very simple to calculate:

$$C_V = \frac{1}{V}\left.\frac{\partial U}{\partial T}\right|_V = \left.\frac{\partial u}{\partial T}\right|_V$$

$$\therefore \quad C_V = D(\varepsilon_F)\frac{\pi^2}{3}k_B^2 T \equiv \gamma T$$

$\gamma$ – Sommerfeld constant

$$\boxed{\gamma = D(\varepsilon_F)\frac{\pi^2}{3}k_B^2 = \frac{n\pi^2}{2\varepsilon_F}k_B^2}$$

## Application of $v_0$ & $C_v$ to calculated parameters:

### Mean free path:

$$\ell = v_0 \tau \simeq v_F \tau$$

### Thermal conductivity:

$$\frac{\kappa}{\sigma T} = \frac{1}{3} v_0^2 \tau C_v$$

$$\therefore \frac{\kappa}{\sigma T} = \frac{\pi^2}{3}\left(\frac{k_B}{e}\right)^2 \simeq 2.4 \times 10^{-8}\ \Omega \cdot \Omega \cdot K^{-2}$$

good agreement $\overline{w}$ exp. (Table 1.6)

### Thermopower:

$$Q = \frac{-\pi^2}{6}\frac{k_B}{e}\left(\frac{k_B T}{\varepsilon_F}\right) \simeq -1.42\left(\frac{k_B T}{\varepsilon_F}\right) \times 10^{-4}\ \frac{V}{K}$$

Much better agreement than Drude

$\Rightarrow$ smaller than Drude by $\mathcal{O}\left(\frac{k_B T}{\varepsilon_F}\right) \simeq 0.01$

@ room temp.